

# When do Projections Commute?

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Necessary and sufficient conditions for commutativity of two projections in Hilbert space are given through properties of so-called conditional connectives which are derived from the conditional probability operator  $PQP$ . This approach unifies most of the known proofs, provides a few new criteria, and permits certain suggestive interpretations for compound properties of quantum-mechanical systems.

## 1. Introduction

Commutativity of two projections  $P$  and  $Q$  in a complex Hilbert space  $H$  plays an important rôle in the mathematical formulation and physical interpretation of quantum-mechanical systems.  $PQ = QP$  is interpreted as “commensurability” of the properties represented by  $P$  and  $Q$ . This means: On a quantum-mechanical system in a given state, measurement of  $P$  and  $Q$  can be made simultaneously or, a measurement of first  $P$  and then  $Q$  affects any state  $\varphi$  in the same way as does a measurement first of  $Q$  and then of  $P$ :

$$\langle \varphi, QP\varphi \rangle = \langle \varphi, PQ\varphi \rangle.$$

Mathematically speaking, this identity is equivalent to the fact that  $PQ$  is a projection onto the meet of  $P$  and  $Q$ , which in turn means physically that  $PQ (= QP)$  is again a “compound” property of the system.

On the other hand, the meet (which by abuse of language we write  $P \wedge Q$ ) of  $P$  and  $Q$  is uniquely defined even for non-commuting projections. In this case,  $PQ$  is not a projection and a fortiori  $PQ \neq QP$ .  $PQ$  is not even an observable (hermitian operator) in  $H$  and hence is not interpreted in quantum mechanics. The interpretation of  $P \wedge Q$ , however, has been controversial (see Jammer’s book [4], p. 353–361).

The purpose of this paper is to present a somewhat unified approach to commutativity proofs for two projections in Hilbert space. We shall derive necessary and sufficient conditions for commutativity from properties of the so-called “con-

ditional probability operator”  $PQP$  (cf. Bub’s discussion in [3], and the relevant literature quoted there). This observable leads to the introduction of derived connectives  $P \sqcap Q$ ,  $P \sqcup Q$ , and the material quasi-implication  $P \rightarrow Q$ . These connectives allow a reasonable physical interpretation for the meet  $P \wedge Q$  even for non-commuting projections. Most of the following material can be proved in the more general setting of quasimodular orthocomplemented lattices (cf. [9] and [10]). These criteria are rephrasings of known results in terms of the new connectives; only (3.17) below appears to be new.

## 2. Conditional Connectives

Let  $P$  and  $Q$  be projections in a complex Hilbert space  $H$ . Because of the one-to-one correspondence between projections and their ranges, we denote the range of  $P$  by  $P$  as well, so that

$$Px = x \quad \text{and} \quad x \in P$$

have the same meaning.

Let  $E_0(PQP)$  denote the null-space of  $PQP$ :

$$E_0(PQP) = \{x \in H \mid PQPx = 0\}.$$

### 2.1. Definition

The “conditional” connective  $P \sqcap Q$  is (the projection onto) the orthocomplement of  $E_0(PQP)$ :

$$P \sqcap Q = E_0^\perp(PQP),$$

read “ $P$  and then  $Q$ ”.

In other words,  $P \sqcap Q$  is the projection onto the range of  $PQP$ .

It follows from  $\langle x, PQPx \rangle = \|QPx\|^2$  that  $PQP x = 0$  if  $QP x = 0$ . Now the following representation of  $E_0(PQP) = E_0(QP)$  ensues:

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## 2.2. Theorem

For all projections  $P, Q$  in  $H$

$$E_0(PQP) = P^\perp \vee (P \wedge Q^\perp),$$

where  $\vee$  may be replaced by  $+$ .

**Proof.**  $QPx=0$  is equivalent to  $Px \in Q$ , and this is certainly fulfilled for all  $x$  in the subspace on the right-hand side.

Conversely, if  $x$  is such that  $Px \in Q$ , then we see from  $x = (I - P)x + Px$  and  $I - P = P^\perp$  that  $x$  belongs also to the right-hand side of the equality above.

## 2.3. Corollary

$$P \sqcap Q = P \wedge (P^\perp \vee Q).$$

2.3 justifies our reading of  $P \sqcap Q$  as “ $P$  and then  $Q$ ”; for, if we interpret the right-hand side of (2.3) via “classical” connectives, we see that  $P \sqcap Q$  is true iff  $P$  is true and it is true that  $Q$  follows “materially” from  $P$ . It is now clear how to define  $P \sqcup Q$ : “ $P$  or then  $Q$ ”, and a material “quasi-implication” or “conditional implication”  $P \rightarrow Q$ :

## 2.4. Definitions

For projections  $P, Q$  in  $H$  put

$$\begin{aligned} P \sqcup Q &:= (P^\perp \sqcap Q^\perp)^\perp \\ &= E_0(P^\perp Q^\perp P^\perp) \\ &= \{x \mid Q^\perp P^\perp x = 0\} \\ &= P \vee (P^\perp \wedge Q); \end{aligned}$$

$$\begin{aligned} P \rightarrow Q &:= P^\perp \sqcup Q \\ &= E_0(PQ^\perp P) = \{x \mid Q^\perp Px = 0\} \\ &= P^\perp \vee (P \wedge Q). \end{aligned}$$

$P \rightarrow Q$  can be read as follows: “ $P \rightarrow Q$  is true iff either  $P^\perp$  is true or the occurrence of the yes-outcome of  $P$  leaves the system in a state which makes true  $Q$ .” (cf. [2], p. 378).

## 2.5. Corollary

For all projections  $P, Q$  in  $H$  (commuting or not), we have

- (1)  $P \wedge Q \leq P \sqcap Q$ ,
  - (2)  $P \vee Q \geq P \sqcup Q$ ,
  - (3)  $P \wedge Q = P \sqcap (P^\perp \sqcup Q) = Q \sqcap (Q^\perp \sqcup P)$ ,
  - (4)  $P \vee Q = P \sqcup (P^\perp \sqcap Q) = Q \sqcup (Q^\perp \sqcap P)$ ,
- note that  $P \sqcup (P^\perp \sqcap Q) = P + (P^\perp \sqcap Q)$ .

**Proof.** (1) and (2) are clear from (2.3) and (2.4). We prove (4): Since

$$P \leq P \vee Q, \quad P \vee Q - P = (P \vee Q) \wedge P^\perp.$$

(3) follows from (4).

$P \wedge Q$  and  $P \vee Q$  can also be expressed in terms of the spectral measure of the observable  $PQP$ :

## 2.6. Theorem

For all projections  $P, Q$  in  $H$

$$P \wedge Q = E_1(PQP) = E_1(QPQ),$$

where  $E_1$  is the respective projection onto the eigenspace with the eigenvalue 1.

**Proof.**  $x \in E_1(PQP)$  iff  $PQP x = x$ . From  $\|x\|^2 = \langle PQP x, x \rangle = \|QP x\|^2 \leq \|P x\|^2 \leq \|x\|^2$  we see that  $P x = x$ , i.e.  $x \in P$ , and also  $QP x = x$ , and together with  $P x = x$ , that  $Q x = x$ . The converse is evident.

2.5 (3) allows a suggestive reading of the meet  $P \wedge Q$ , whether  $P$  and  $Q$  commute or not:

$P \wedge Q$  iff  $P$  and then  $P$  quasi-implies  $Q$ , which is the same as  $Q$  and then  $Q$  quasi-implies  $P$ .

## 3. Commutativity of Projections

Abbreviate  $P \sim Q$  for  $PQ = QP$ . Obviously,

$$\begin{aligned} P \sim Q &\Leftrightarrow Q \sim P \Leftrightarrow P \sim Q^\perp \\ &\Leftrightarrow P^\perp \sim Q \Leftrightarrow P^\perp \sim Q^\perp. \end{aligned} \quad (3.1)$$

**Main Theorem:**  $P \sim Q$  is equivalent to each of the following equalities or inequalities in (3.2) through (3.17). (We shall prove only sufficiency; the proof that  $P \sim Q$  implies (3.2) through (3.17) is straightforward and will be omitted).

$$P \wedge Q = P \sqcap Q. \quad (3.2)$$

**Proof:** (3.2) is the same as

$$E_1(PQP) = E_0^\perp(PQP),$$

which means that  $PQP$  is the projection onto  $P \wedge Q$ . Because of  $E_0(PQP) = E_0(QP)$ ,  $PQP$  and  $QP$  coincide on  $(P \wedge Q)^\perp$ . On the other hand, for every  $x \in P \wedge Q$ ,  $x = QPx$  and  $x = PQPx$ , so that  $PQP$  and  $QP$  are identical.

$$P \vee Q = P \sqcup Q. \quad (3.3)$$

**Proof:** (3.2) and (3.1).

$$P = P \wedge Q + P \wedge Q^\perp. \quad (3.4)$$

Proof:  $P \wedge Q^\perp \leq P$  implies  $P \wedge Q = P - P \wedge Q^\perp = P \wedge (P \wedge Q^\perp)^\perp = P \wedge (P^\perp \vee Q) = P \cap Q$ . Now apply (3.2).

(3.2) is often used to define the 2-place relation  $C$  of commensurability

$$(P, Q) \in C \Leftrightarrow P = P \wedge Q + P \wedge Q^\perp$$

in an orthocomplemented lattice  $L_0$  (e.g. Mittelstaedt [9], p. 32). It is worth noting that  $C$  is symmetrical if and only if  $L_0$  is quasimodular (some authors say “orthomodular” or “weakly modular”):

$$P \leq Q \Rightarrow Q \cap P = P.$$

If this implication holds in  $L_0$ , (3.1) is true in  $L_0$  (see Mittelstaedt [9], p. 30–34); the reverse implication  $Q \cap P = P \Rightarrow P \leq Q$  is always true. If equality of the antecedent is weakened to mere inclusion, we get that  $P \sim Q$  is a consequence of

$$Q \cap P \leq P. \quad (3.5)$$

Proof:  $P \wedge Q \leq Q \cap P$  from (2.5) (1);  $Q \cap P \leq Q$  together with (3.5) and (3.2) give  $P \sim Q$ .

$$P \leq Q \sqcup P. \quad (3.6)$$

Proof: (3.5) and (3.1).

$$P \leq (Q \rightarrow P). \quad (3.7)$$

Proof:  $Q \rightarrow P = Q^\perp \sqcup P \geq P$ . Apply (3.6) and (3.1). (We remark that always  $P \sim (P \rightarrow Q)$ ; cf. [9], p. 41, 2.40(b)).

$$P \rightarrow (Q \rightarrow P) = H. \quad (3.8)$$

Proof: (3.8) is the same as saying

$$P^\perp \sqcup (Q^\perp \sqcup P) = H.$$

Because of (2.5) (2) we have also  $H = P^\perp \vee (Q^\perp \sqcup P)$ . But then  $P \leq H$  implies  $P \leq Q^\perp \sqcup P$ , and (3.6) together with (3.1) yield  $P \sim Q$ .

$$P \cap Q = Q \cap P. \quad (3.9)$$

Proof: Using (3.2), we have to prove that (3.9) implies  $P \cap Q = P \wedge Q$ . We know that always  $P \cap Q \geq P \wedge Q$ . On the other hand, if  $x \in P \cap Q = Q \cap P$ , then  $x \in P$  and  $x \in Q$  (from the representation (2.3)), i.e.  $x \in P \wedge Q$ .

$$P \sqcup Q = Q \sqcup P. \quad (3.10)$$

Proof: (3.9) and (3.1).

$$P \rightarrow Q = Q^\perp \rightarrow P^\perp. \quad (3.11)$$

Proof: (3.11) says  $P^\perp \sqcup Q = Q \sqcup P^\perp$ . Apply (3.10) and (3.1).

$$w_\varphi(P \vee Q) \leq w_\varphi(P) + w_\varphi(Q) \quad (3.12)$$

for all states  $\varphi \in H$ , where  $w_\varphi(R) := \langle \varphi, R\varphi \rangle$  for projections  $R$  in  $H$ .

Proof:  $P \vee Q = P + (P^\perp \cap Q)$  from (2.5) (4), and  $w_\varphi(P^\perp \cap Q) \leq w_\varphi(Q)$  for all states  $\varphi$  if and only if  $P^\perp \cap Q \leq Q$ . Using (3.5) and (3.1) gives  $P \sim Q$ .

(Cf. also Jauch’s lemma, p. 117 of [5]).

$$P = Q \cap P + Q^\perp \cap P. \quad (3.13)$$

Proof:

$$\begin{aligned} Q \cap P + Q^\perp \cap P &= \{x \mid PQx = 0\}^\perp \\ &\quad + \{x \mid PQ^\perp x = 0\}^\perp \\ &= \{x \mid PQx = 0 \text{ and } \\ &\quad Px - PQx = 0\}^\perp \\ &= \{x \mid Px = 0 \text{ and } \\ &\quad PQx = 0\}^\perp \\ &= (P^\perp \wedge Q^\perp \sqcup P^\perp)^\perp \\ &= P \vee (Q \cap P) \\ &= P + P^\perp \cap (Q \cap P). \end{aligned}$$

Therefore, (3.13) is equivalent to  $P^\perp \cap (Q \cap P) = 0$  or  $H = P \sqcup (Q^\perp \sqcup P^\perp) = P^\perp \rightarrow (Q \rightarrow P^\perp)$ , which by (3.8) and (3.1) implies  $P \sim Q$ .

The proof of (3.13) shows also that  $P^\perp \cap (Q \cap P) = P^\perp \cap (Q^\perp \cap P) =: I(P, Q)$ , which may be called the “interference term”, and (3.13) is equivalent to

$$I(P, Q) = 0. \quad (3.14)$$

$$P = QPQ + Q^\perp PQ^\perp. \quad (3.15)$$

Proof: Due to (3.13), (3.15) is the same as

$$QPQ + Q^\perp PQ^\perp = Q \cap P + Q^\perp \cap P.$$

Since always  $QPQ \leq Q \cap P$  and  $Q^\perp PQ^\perp \leq Q^\perp \cap P$ , the latter equality can only hold if  $QPQ = Q \cap P$  and  $Q^\perp PQ^\perp = Q^\perp \cap P$ . But this means that  $QPQ$  is the projection onto  $Q \wedge P$ , and thus coincides with  $PQ$  as was shown in the proof of (3.2).

From  $QPQ + Q^\perp PQ^\perp = P - (QPQ^\perp + Q^\perp PQ)$  it can be seen that (3.15) holds if and only if  $J(P, Q) := QPQ^\perp + Q^\perp PQ$  is zero:

$$J(P, Q) = 0. \quad (3.16)$$

$J(P, Q)$  is the observable which defines Mittelstaedt’s probability of interference ([8], p. 215):

$$w_\varphi^{\text{int}}(P, Q) = \langle \varphi, J(P, Q)\varphi \rangle,$$

which is zero if and only if  $(J(P, Q)$  is hermitian!) the condition (3.16) holds, i.e. iff  $P \sim Q$ .

(3.4), (3.13) and (3.15) are saying that each of the following representations for  $P$  is equivalent to  $P \sim Q$ :

$$\begin{aligned} P &= P \wedge Q + P \wedge Q^\perp, \\ P &= Q \sqcap P + Q^\perp \sqcap P, \\ P &= QPQ + Q^\perp P Q^\perp. \end{aligned}$$

As our final criterium we show that  $P \sim Q$  is equivalent to

$$PQP = QPQ. \quad (3.17)$$

Proof: By (3.2),  $P \sim Q$  iff  $P \wedge Q = P \sqcap Q$ , i.e.  $E_1(PQP) = E_0(PQP)$ . This equality is true iff  $PQP$  is a projection. But, using (3.17) twice, we get

$$\begin{aligned} (PQP)^2 &= PQPPQP = PQ(PQP) \\ &= PQ(QPQ) = P(QPQ) \\ &= P(PQP) = PQP, \end{aligned}$$

so that the hermitian operator  $PQP$  is idempotent, i.e. in fact a projection.

From the standpoint of physical interpretation, (3.17) is to be expected:  $PQP$  is the defining operator for the "joint" probability of  $P$  and (then)  $Q$  and determines the conditional probability of  $Q$ , given  $P$ . Considering this interpretation, and (3.16), for instance, which is equivalent to (3.17), it comes as no surprise that  $PQP = QPQ$  should imply  $P \sim Q$ . Mathematically speaking, however, this implication seems curious: (3.17) means that for  $PQ = QP$  it is sufficient that  $PQ$  has the same value for  $Px$  as  $QP$  has for  $Qx$ , for all  $x \in H$ . In other words, (3.17) permits an implication from the equality of positive self-adjoint operators  $PQP$  and  $QPQ$  to the equality of *prima facie* more general operators  $PQ$  and  $QP$ . Putting  $A = PQ$ ,  $A^* = QP$ , (3.17) may be restated as:  $A = A^*$  is equivalent to  $AA^* = A^*A$ , i.e. for  $A = PQ$  self-adjointness and normality are the same.

For this reason it may be of interest to have a proof of (3.17) independent of (3.2) and of the representation of  $P \sqcap Q$  and  $P \wedge Q$  through the spectral measure of  $PQP$ . We shall do so now.

*Proposition.* For any two projections  $P$  and  $Q$  in a complex Hilbert space  $H$ , the commutativity relation  $PQ = QP$  is equivalent to  $PQP = QPQ$ .

Again, we need only prove the non-trivial direction.

We need the following

Lemma: Let  $A$  and  $B$  be bounded linear operators in  $H$  such that

- (1)  $AB = BA$ ,
- (2)  $A^2 = B^2$ , and
- (3)  $(A - B) = -(A - B)^*$ .

Then (4)  $E$  commutes with any transformation that commutes with  $A - B$ , and (5)  $A = (2E - I)B$ , where  $E$  is the orthogonal projection onto the null-space  $M = E_0(A - B)$  of  $A - B$ .

Proof of the Lemma: (Cf. [1], p. 424, Theorem 23.3; note that in our proof  $A$  and  $B$  need not be self-adjoint!). Suppose that  $C$  commutes with  $A - B$ . This implies  $C(M) \subset M$ . From

$$\begin{aligned} C(A - B) &= (A - B)C \Rightarrow (A - B)^* C^* \\ &= C^*(A - B)^* \end{aligned}$$

and (3), we have

$$(A - B)C^* = C^*(A - B),$$

which implies  $C^*(M) \subset M$ . Therefore  $C$  reduces  $M$ , i.e.  $CE = EC$ , proving (4).

From (1) and (2) we have

$$(A - B)(A + B) = A^2 - B^2 = 0,$$

i.e. (6)

$$E(A + B) = A + B.$$

For any vector  $z \in H$  write  $z = x + y$ , where  $x \in M$  and  $y \in M^\perp$ . It follows

$$E(A - B)z = E(A - B)x + E(A - B)y.$$

The first term on the right is zero, because  $x \in M = E_0(A - B)$  and the second is zero because  $E$  commutes with  $A - B$ , according to (4).

Hence

$$(7) \quad E(A - B) = 0.$$

Combining (6) and (7), gives

$$E(A + B) - E(A - B) = A + B$$

or

$$A = 2 \cdot EB - B = (2E - I)B,$$

which proves (5).

We wish to apply the Lemma for  $A=PQ$ ,  $B=QP$ . Assumption (1) is the same as  $PQP=QPQ$ . Using this, and observing

$$(8) \quad (PQ)^2 = PQPQ = PPQP = PQP \\ = QPQ = (QP)^2,$$

we note that (2) is fulfilled.

Moreover,

$$(PQ - QP)^* = QP - PQ \\ = -(PQ - QP),$$

which is assumption (3) of our Lemma.

**Proof of the Proposition:** From

$$(PQP)^2 = PQPPQP = PQPQP \\ = PQQPQ = (PQ)^2$$

and (8), we see that  $PQP=QPQ$  must be a projection. We claim that  $PQP$  is the projection onto  $P \wedge Q$ . This may be seen from

$$PQP = (PQP)^2 = (PQ)^2,$$

i.e.

$$PQP = (PQ)^{2k}, \quad k \geq 1,$$

and from the fact that the projection onto  $P \wedge Q$  is given by the limit of  $(PQ)^n$ ,  $n \rightarrow \infty$ .

We prove  $PQ = QP$ .

If  $z \in P \wedge Q$ , trivially  $PQz - QPz = z - z = 0$ . If  $z \in P^\perp \vee Q^\perp$ , write  $z = \lim (x_n + y_n)$ , where  $x_n \in P^\perp$  and  $y_n \in Q^\perp$ . Using (5) of our Lemma gives

$$PQ = (2E - I)QP \quad \text{and} \\ QP = (2E - I)PQ,$$

where  $E$  is the orthogonal projection onto

$$E_0(PQ - QP) = E_0(QP - PQ)$$

(note that the assumptions of the Lemma are symmetrical in  $A, B$ ).

Therefore, using continuity,

$$PQz = \lim_n (PQx_n + PQy_n) \\ = \lim_n PQx_n = \lim_n (2E - I)QPx_n = 0,$$

similarly  $QPz = 0$ .

Hence,  $PQ$  and  $QP$  coincide on  $H$ .

*Remark*

It should be noted that the Proposition is also a special case of a rather deep theorem by Fuglede-Putnam, Rosenblum (cf. [11], p. 300, Theorem 12.16, where Rosenblum's proof is given):

Assume that  $A, B, T$  are bounded transformations on  $H$ ,  $A$  and  $B$  are normal, and

$$AT = TB.$$

Then  $A^*T = TB^*$ .

Taking  $A=PQ$ ,  $B=QP$ ,  $T=P$  yields our Proposition.

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